Synchronisation of a Class of Nonlinear Time-Delay Systems Applied to Two Unidirectionally Coupled External Cavity Laser Diodes

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Abstract. This paper shows the partial-state synchronisation of a class of nonlinear time-delay systems by means of an approximating sequence technique and optimal control. The approximating sequence technique reduces a nonlinear time-delay system into a sequence of linear time-varying (LTV) equations, in order to approximate to the solution of the original nonlinear system. In each of these equations, the linear optimal regulator problem is utilised in order to minimise the error between the drive system and the response system. Once the error signal is close enough to zero, then the drive and response systems are synchronous. The procedure is employed in the synchronisation of two unidirectionally coupled chaotic external cavity laser diodes using numerical simulations.

1 Introduction

The synchronisation of chaotic systems is a subject that has been receiving a lot of attention of researchers throughout many years. Since the seminal paper of Pecora and Carroll [1], the chaotic synchronisation research field has been intensively studied due to its potential use for secure communications. Synchronised chaotic systems might be identical or different, the coupling between them may be unidirectional (drive-response or master-slave coupling) or bi-directional (mutual coupling) and the driving force can be deterministic or stochastic.

In the field of lasers, many research groups have focused their attention on understanding the chaotic synchronisation phenomena in unidirectionally coupled lasers and its potential utilisation in secure communications. The external cavity laser diodes or semiconductor lasers subject to optical feedback present a diversity of behaviours [2]. The reflected light from the optical feedback might create chaos in the laser dynamics. Hence, a lot of research is concentrated on suppressing this kind of dynamics. Nevertheless, current studies are taking advantage of these chaotic dynamics in order to employ them for hiding messages and then, by synchronising the oscillations, the messages can be extracted.

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Semiconductor lasers with optical feedback can be modelled by delay differential equations. A well-known theoretical description of the system is the Lang-Kobayashi model [3], which provides an effective representation of the dynamics produced by semiconductor lasers optically coupled with a distant reflector.

Mirasso, Colet et al. [4] numerically demonstrated the synchronisation of two chaotic semiconductor lasers with optical feedback when a small amount of output intensity from one is injected into the other. They also showed the codification and decodification of a message in a chaotic carrier. Sivaprakasam and Shore [5] were the first to demonstrate experimentally the synchronisation of two unidirectionally coupled chaotic external cavity laser diodes.

The numerical analyses of synchronisation of semiconductor lasers with optical an optoelectronic feedback have been studied in [6, 7, 8], and references therein. In [6], the synchronisation has been obtained provided the operation conditions are adequate and the parameter regions in which synchronisation is achieved are presented. In [7], complete and time lag synchronisations in unidirectionally coupled semiconductor lasers are studied, in which it is shown how the degree of synchronisation decreases with mismatch in the laser parameters between the transmitter and receiver. In [8], it is demonstrated that the coupling strength between drive and response influences the quality of synchronisation.

Recent works on synchronisation of time-delay systems are: [9] applies the Generalized Hamiltonian forms and observer approach to synchronise time-delay-feedback Chua's circuits in order to transmit encrypted signals, [10] uses adaptive control theory for the stabilisation and synchronisation of chaotic Lur'e systems with time-varying delay, and [11] proposes a crypto-system based on a chaotic time delay model that uses the synchronisation of chaotic signals which is executed through a nonlinear state observer design.

In this paper, the partial-state synchronisation of a class of nonlinear time-delay systems is presented. The method uses an approximating sequence technique [12] that approximates to the solution of nonlinear systems by producing a sequence of LTV equations; then in each of these equations, the linear optimal regulator problem is applied in order to minimise the error between the drive system and the response system. After the error signal is minimised, the signals of the drive and response systems are synchronised. The proposed method is applied numerically to the synchronisation of two unidirectionally coupled chaotic external cavity laser diodes, in which the coupling between the drive and response can be observed in the controller. This controller contains the error signal between the drive and response. The advantage of this method is that there is no need to adjust the coupling strength between the drive and response, and it is not required to tune the feedback strengths of the external cavities of each laser. In addition, the parameter mismatches between the drive and response are compensated for, consequently, it is not a requirement that the lasers are identical. Also, it is not necessary to use parameter regions in which synchronisation occurs because the method works in all operating points of the phase space.

This paper is divided as follows: in Sect.2, the approximating sequence technique is briefly introduced. The procedure of the synchronisation of nonlinear time-delay systems using the approximating sequence technique and linear optimal control is shown in Sect.3. The application of the method for synchronising semiconductor lasers subject to optical feedback is described in Sect.4. Finally, conclusions are given in Sect.5.

2 The Approximating Sequence Technique

This section recalls the method that approximates to the solution of nonlinear systems [12, 13, 14]. Given a nonlinear system of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t) \ , \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \ , \end{aligned} \tag{1}$$

where $\mathbf{x}(t)$ is a n-dimensional state vector, $\mathbf{A}\in\mathbb{R}^{n\times n}$ and $\mathbf{B}\in\mathbb{R}^{n\times m}$ are nonlinear matrix-valued functions of $\mathbf{x}(t)$ which are locally Lipschitz, and $\mathbf{u}(t)$ is a m-dimensional unconstrained control vector.

The above system can be replaced by a sequence of LTV approximations

$$\begin{aligned} \dot{\mathbf{x}}^{[1]}(t) &= \mathbf{A}(\mathbf{x}_0)\mathbf{x}^{[1]}(t) + \mathbf{B}(\mathbf{x}_0)\mathbf{u}^{[1]}(t) \ , \\ &\vdots \\ \dot{\mathbf{x}}^{[i]}(t) &= \mathbf{A}\left(\mathbf{x}^{[i-1]}(t)\right)\mathbf{x}^{[i]}(t) + \mathbf{B}\left(\mathbf{x}^{[i-1]}(t)\right)\mathbf{u}^{[i]}(t) \ , \end{aligned} \tag{2}$$

where the initial conditions are: $\mathbf{x}^{[1]}(t_0)=\mathbf{x}^{[2]}(t_0)=\ldots=\mathbf{x}^{[i]}(t_0)=\mathbf{x}_0$.

The solution of each of the approximations converges to the solution of the original nonlinear system.

Since the sequence of approximations is formed by linear differential equations, then linear control techniques can be implemented to each of the equations. That is, the control of nonlinear systems can be approached by linear control techniques through applying the approximating sequence technique. This is providing that the nonlinear system fulfils the local Lipschitz requirement.

3 Synchronisation of Nonlinear Time-Delay Systems

This section shows the synchronisation between two nonlinear time-delay systems of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x}(t),\mathbf{x}(t-\tau))\mathbf{x}(t) + \mathbf{A}_\tau(\mathbf{x}(t),\mathbf{x}(t-\tau))\mathbf{x}(t-\tau) + \mathbf{z}(t) \ , \\ \mathbf{x}(t) &= \boldsymbol{\xi}(t), \quad t\in[t_0-\tau, t_0) \ , \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \ , \end{aligned} \tag{3}$$

where $\mathbf{x}(t)\in\mathbb{R}^n$ is a state vector, $\mathbf{A}\in\mathbb{R}^{n\times n}$ and $\mathbf{A}_\tau\in\mathbb{R}^{n\times n}$ are nonlinear matrix-valued functions of $\mathbf{x}(t)$ which are locally Lipschitz, $\mathbf{z}(t)\in\mathbb{R}^n$ is a forcing vector, τ is the time delay, a positive constant, and $\boldsymbol{\xi}(t)$ is a history function.

The synchronisation is accomplished by utilising the approximating sequence technique in combination with the linear optimal regulator problem. The procedure is explained as follows.

Consider a nonhomogeneous nonlinear time-delay system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}_\mathrm{d}(\mathbf{x}(t),\mathbf{x}(t-\tau))\mathbf{x}(t) + \mathbf{A}_{\tau_\mathrm{d}}(\mathbf{x}(t),\mathbf{x}(t-\tau))\mathbf{x}(t-\tau) + \mathbf{z}_\mathrm{d}(t) \ , \\ \mathbf{x}(t) &= \boldsymbol{\xi}_\mathrm{d}(t), \quad t \in [t_0-\tau, t_0) \ , \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \ ,\end{aligned} \tag{4}$$

as the drive system. The corresponding response system is

$$\begin{aligned}\dot{\mathbf{y}}(t) &= \mathbf{A}_\mathrm{r}(\mathbf{y}(t),\mathbf{y}(t-\tau))\mathbf{y}(t) + \mathbf{A}_{\tau_\mathrm{r}}(\mathbf{y}(t),\mathbf{y}(t-\tau))\mathbf{y}(t-\tau) \\ &\quad + \mathbf{B}(\mathbf{y}(t),\mathbf{y}(t-\tau))\mathbf{u}(t) + \mathbf{z}_\mathrm{r}(t) \ , \\ \mathbf{y}(t) &= \boldsymbol{\xi}_\mathrm{r}(t), \quad t \in [t_0-\tau, t_0) \ , \\ \mathbf{y}(t_0) &= \mathbf{y}_0 \ ,\end{aligned} \tag{5}$$

where $\mathbf{B}\in\mathbb{R}^{n\times m}$ and $\mathbf{u}(t)\in\mathbb{R}^m$ is an unconstrained control vector. The term $\mathbf{B}(\mathbf{y}(t),\mathbf{y}(t-\tau))\mathbf{u}(t)$ is added to the response system in order to minimise the error given by $\mathbf{e}(t)=\mathbf{x}(t)-\mathbf{y}(t)$. Once $\mathbf{e}(t)$ is minimised, then $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are synchronous.

By differentiating the error and by simplifying, the next equation is created

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}_\mathrm{r}(\psi(t))\mathbf{e}(t) + \mathbf{A}_{\tau_\mathrm{r}}(\psi(t))\mathbf{e}(t-\tau) + \big(\mathbf{A}_\mathrm{d}(\chi(t)) - \mathbf{A}_\mathrm{r}(\psi(t))\big)\mathbf{x}(t) + \\ &\quad \big(\mathbf{A}_{\tau_\mathrm{d}}(\chi(t)) - \mathbf{A}_{\tau_\mathrm{r}}(\psi(t))\big)\mathbf{x}(t-\tau) - \mathbf{B}(\psi(t))\mathbf{u}(t) + \mathbf{z}_\mathrm{d}(t) - \mathbf{z}_\mathrm{r}(t) \ , \\ \mathbf{e}(t) &= \boldsymbol{\xi}_\mathrm{e}(t), \quad t \in [t_0-\tau, t_0) \ , \\ \mathbf{e}(t_0) &= \mathbf{e}_0 \ .\end{aligned} \tag{6}$$

where $(\chi(t))\triangleq(\mathbf{x}(t),\mathbf{x}(t-\tau))$ and $(\psi(t))\triangleq(\mathbf{y}(t),\mathbf{y}(t-\tau))$.

Decomposing (6) into a sequence of LTV equations produces

$$\begin{aligned}\dot{\mathbf{e}}^{[1]}(t) &= \mathbf{A}_\mathrm{r}(\psi_0)\mathbf{e}^{[1]}(t) + \mathbf{A}_{\tau_\mathrm{r}}(\psi_0)\mathbf{e}^{[1]}(t-\tau) + \big(\mathbf{A}_\mathrm{d}(\chi_0) - \mathbf{A}_\mathrm{r}(\psi_0)\big)\mathbf{x}^{[1]}(t) \quad (7) \\ &\quad + \big(\mathbf{A}_{\tau_\mathrm{d}}(\chi_0) - \mathbf{A}_{\tau_\mathrm{r}}(\psi_0)\big)\mathbf{x}^{[1]}(t-\tau) - \mathbf{B}(\psi_0)\mathbf{u}^{[1]}(t) + \mathbf{z}_\mathrm{d}(t) - \mathbf{z}_\mathrm{r}(t) \ , \\ \dot{\mathbf{e}}^{[i]}(t) &= \mathbf{A}_\mathrm{r}\Big(\psi^{[i-1]}(t)\Big)\mathbf{e}^{[i]}(t) + \mathbf{A}_{\tau_\mathrm{r}}\Big(\psi^{[i-1]}(t)\Big)\mathbf{e}^{[i]}(t-\tau) + \bigg(\mathbf{A}_\mathrm{d}\Big(\chi^{[i-1]}(t)\Big) \\ &\quad - \mathbf{A}_\mathrm{r}\Big(\psi^{[i-1]}(t)\Big)\bigg)\mathbf{x}^{[i]}(t) + \bigg(\mathbf{A}_{\tau_\mathrm{d}}\Big(\chi^{[i-1]}(t)\Big) - \\ &\quad \mathbf{A}_{\tau_\mathrm{r}}\Big(\psi^{[i-1]}(t)\Big)\bigg)\mathbf{x}^{[i]}(t-\tau) - \mathbf{B}\Big(\psi^{[i-1]}(t)\Big)\mathbf{u}^{[i]}(t) + \mathbf{z}_\mathrm{d}(t) - \mathbf{z}_\mathrm{r}(t) \ .\end{aligned}$$

where $(\chi_0)\triangleq(\mathbf{x}_0,\mathbf{x}_0)$ and $(\psi_0)\triangleq(\mathbf{y}_0,\mathbf{y}_0)$, $\big(\chi^{[i-1]}(t)\big)\triangleq\big(\mathbf{x}^{[i-1]}(t),\mathbf{x}^{[i-1]}(t-\tau)\big)$ and $\big(\psi^{[i-1]}(t)\big)\triangleq\big(\mathbf{y}^{[i-1]}(t),\mathbf{y}^{[i-1]}(t-\tau)\big)$.

The first approximation is chosen as a linear time-invariant equation. By applying the approximating sequence technique, the drive system $\big(\dot{\mathbf{x}}(t)\big)$ in (4) can be reduced into a sequence of LTV approximations. The solution of the first approximation, $\mathbf{x}^{[1]}(t)$, is replaced in $\dot{\mathbf{e}}^{[1]}(t)$ in order to obtain the solution of the first approximation $\mathbf{e}^{[1]}(t)$. Then, for the subsequent approximation, the solution of $\mathbf{y}^{[1]}(t)$ from $\mathbf{e}^{[1]}(t)=\mathbf{x}^{[1]}(t)-\mathbf{y}^{[1]}(t)$ must be calculated. With the solutions of the first approximation $(\mathbf{x}^{[1]}(t)$ and $\mathbf{y}^{[1]}(t))$ and the solution $\mathbf{x}^{[2]}(t)$, the approximation $\mathbf{e}^{[2]}(t)$ can be found, and then $\mathbf{y}^{[2]}(t)$ is obtained. The same procedure follows for obtaining the solution of the ith approximation $\mathbf{e}^{[i]}(t)$. The next technique shows how the controller $\mathbf{u}^{[i]}(t)$ is designed.

Consider the ith approximation

$$\begin{aligned}\dot{\mathbf{e}}(t) = \mathbf{A}_{\mathrm{r}}(T)\mathbf{e}(t) + \mathbf{A}_{\tau_{\mathrm{r}}}(T)\mathbf{e}(t-\tau) + \big(\mathbf{A}_{\mathrm{d}}(T) - \mathbf{A}_{\mathrm{r}}(T)\big)\mathbf{x}(t) + \\ \big(\mathbf{A}_{\tau_{\mathrm{d}}}(T) - \mathbf{A}_{\tau_{\mathrm{r}}}(T)\big)\mathbf{x}(t-\tau) - \mathbf{B}(T)\mathbf{u}(t) + \mathbf{z}_{\mathrm{d}}(t) - \mathbf{z}_{\mathrm{r}}(t) \ ,\end{aligned} \tag{8}$$

where $(t, t-\tau)\triangleq(T)$. This equation is a LTV time-delay equation. Note that at the ith iteration, the approximations $\big(\mathbf{x}^{[i-1]}(t), \mathbf{x}^{[i-1]}(t-\tau)\big)$ and $\big(\mathbf{y}^{[i-1]}(t), \mathbf{y}^{[i-1]}(t-\tau)\big)$ are known, hence the equations in (7) are all of the form (8).

The finite-time linear quadratic cost functional

$$J = \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{F}(t_{\mathrm{f}})\mathbf{e}(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} \Big\{\mathbf{e}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)\Big\}dt \ , \tag{9}$$

together with (8) minimise the error $\mathbf{e}(t)=\mathbf{x}(t)-\mathbf{y}(t)$. Where the terminal state $\mathbf{e}(t_f)$ is required to be as close as possible to the zero state. $\mathbf{F}\in\mathbb{R}^{n\times n}$ and $\mathbf{Q}\in\mathbb{R}^{n\times n}$ are symmetric positive semi-definite and $\mathbf{R}\in\mathbb{R}^{m\times m}$ is symmetric positive definite. The problem of determining an input $\mathbf{u}(t)$, $t_0\le t\le t_f$, for which the criterion (9) is minimal is called the Linear Optimal Regulator Problem.

Using (8) and the cost function (9), the Hamiltonian for this problem is

$$\begin{aligned}\mathbf{H}\big(\mathbf{e}(t), \mathbf{e}(t-\tau), \mathbf{u}(t), \boldsymbol{\lambda}(t), t, t-\tau\big) = \frac{1}{2}\Big[\mathbf{e}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{e}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)\Big] + \\ \boldsymbol{\lambda}^{\mathrm{T}}(t)\Big\{\mathbf{A}_{\mathrm{r}}(T)\mathbf{e}(t) + \mathbf{A}_{\tau_{\mathrm{r}}}(T)\mathbf{e}(t-\tau) + \big(\mathbf{A}_{\mathrm{d}}(T) - \mathbf{A}_{\mathrm{r}}(T)\big)\mathbf{x}(t) + \\ \big(\mathbf{A}_{\tau_{\mathrm{d}}}(T) - \mathbf{A}_{\tau_{\mathrm{r}}}(T)\big)\mathbf{x}(t-\tau) - \mathbf{B}(T)\mathbf{u}(t) + \mathbf{z}_{\mathrm{d}}(t) - \mathbf{z}_{\mathrm{r}}(t)\Big\} \ .\end{aligned} \tag{10}$$

The optimality conditions are:

$$\dot{\boldsymbol{\lambda}}(t) = \begin{cases} -\frac{\partial \mathbf{H}}{\partial \mathbf{e}} - \frac{\partial \mathbf{H}(t')}{\partial \mathbf{e}(t-\tau)}\Big|_{t'=t+\tau} = -\mathbf{Q}(t)\mathbf{e}(t) - \mathbf{A}_{\mathrm{r}}^{\mathrm{T}}(T)\boldsymbol{\lambda}(t) - \\ \qquad \mathbf{A}_{\tau_{\mathrm{r}}}(T)\boldsymbol{\lambda}(t+\tau), \quad t_0 \le t < t_{\mathrm{f}} - \tau \ , \\ -\frac{\partial \mathbf{H}}{\partial \mathbf{e}} = -\mathbf{Q}(t)\mathbf{e}(t) - \mathbf{A}_{\mathrm{r}}^{\mathrm{T}}(T)\boldsymbol{\lambda}(t) \qquad t_{\mathrm{f}} - \tau \le t \le t_{\mathrm{f}} \ . \end{cases} \tag{11}$$

$$\boldsymbol{\lambda}(t_{\mathrm{f}}) = \frac{\partial J}{\partial \mathbf{e}(t_{\mathrm{f}})} = \mathbf{F}(t_{\mathrm{f}})\mathbf{e}(t_{\mathrm{f}}) \ , \tag{12}$$

$$\frac{\partial \mathbf{H}}{\partial \mathbf{u}} = 0 = \mathbf{R}(t)\mathbf{u}(t) - \boldsymbol{\lambda}^{\mathrm{T}}(t)\mathbf{B}(T) \ . \tag{13}$$

The optimal control law, which can be obtained from (13) is

$$\mathbf{u}(t) = \mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(T)\boldsymbol{\lambda}(t) \ . \tag{14}$$

The adjoint vector is defined as

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{e}(t) + \mathbf{g}(t) \ , \tag{15}$$

which is used in order to decouple equations (11), (14) and (8). $\mathbf{P}(t) \in \mathbb{R}^{n \times n}$, is symmetric and positive-definite and is the solution of the matrix Riccati differential equation. $\mathbf{g}(t) \in \mathbb{R}^n$ is used for compensating for the dynamics produced by the time-delay term, the forcing terms and the LTV terms.

After substituting (15) in (14) the controller is produced

$$\mathbf{u}(t) = \mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(T)\big(\mathbf{P}(t)\mathbf{e}(t) + \mathbf{g}(t)\big) \ . \tag{16}$$

By taking the derivative of (15) and by substituting (8) and (16) into the result, the equation can then be equated to each case of the costate equation (11) after having substituted (15) into each of them. This yields the Riccati equation

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{B}(T)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(T)\mathbf{P}(t) - \mathbf{Q}(t) - \mathbf{A}_{\mathrm{r}}^{\mathrm{T}}(T)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{A}_{\mathrm{r}}(T) \ , \tag{17}$$

which can be calculated for the time span $[t_0, t_f]$, and the compensating vector

$$\begin{aligned}\dot{\mathbf{g}}(t) = {} & \Big(\mathbf{P}(t)\mathbf{B}(T)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(T) - \mathbf{A}_{\mathrm{r}}^{\mathrm{T}}(T)\Big)\mathbf{g}(t) - \mathbf{P}(t)\mathbf{A}_{\tau_{\mathrm{r}}}(T)\mathbf{e}(t-\tau) - \\ & \mathbf{P}(t)\mathbf{z}_{\mathrm{d}}(t) + \mathbf{P}(t)\mathbf{z}_{\mathrm{r}}(t) - \mathbf{A}_{\tau_{\mathrm{r}}}^{\mathrm{T}}(T+\tau)\Big(\mathbf{P}(t+\tau)\mathbf{e}(t+\tau) + \mathbf{g}(t+\tau)\Big) \\ & - \mathbf{P}(t)\Big[\big(\mathbf{A}_{\mathrm{d}}(T) - \mathbf{A}_{\mathrm{r}}(T)\big)\mathbf{x}(t) + \big(\mathbf{A}_{\tau_{\mathrm{d}}}(T) - \mathbf{A}_{\tau_{\mathrm{r}}}(T)\big)\mathbf{x}(t-\tau)\Big] \ , \\ & \qquad\qquad \text{for} \quad t_0 \le t < t_f - \tau \ ,\end{aligned} \tag{18a}$$

$$\begin{aligned}\dot{\mathbf{g}}(t) = {} & \Big(\mathbf{P}(t)\mathbf{B}(T)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(T) - \mathbf{A}_{\mathrm{r}}^{\mathrm{T}}(T)\Big)\mathbf{g}(t) - \mathbf{P}(t)\mathbf{A}_{\tau_{\mathrm{r}}}(T)\mathbf{e}(t-\tau) \\ & - \mathbf{P}(t)\mathbf{z}_{\mathrm{d}}(t) + \mathbf{P}(t)\mathbf{z}_{\mathrm{r}}(t) - \mathbf{P}(t)\Big[\big(\mathbf{A}_{\mathrm{d}}(T) - \mathbf{A}_{\mathrm{r}}(T)\big)\mathbf{x}(t) + \\ & \big(\mathbf{A}_{\tau_{\mathrm{d}}}(T) - \mathbf{A}_{\tau_{\mathrm{r}}}(T)\big)\mathbf{x}(t-\tau)\Big] \ , \qquad \text{for} \quad t_f - \tau \le t \le t_f \ ,\end{aligned} \tag{18b}$$

where $(t+\tau, t) \triangleq (T+\tau)$.

Equations (17) and (18) can be solved by numerically integrating them backwards in time, from t_f to t_0, starting from the final conditions $\mathbf{P}(t_f)=\mathbf{F}(t_f)$ and $\mathbf{g}(t_f)=0$. These terminal conditions are found by equating (12) and (15) at t_f.

Once $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are calculated, then the results are substituted into (16) in order to produce the controller. After finding $\mathbf{u}(t)$, then this is substituted in (8) for producing the minimised error signal

$$\begin{aligned}\dot{\mathbf{e}}(t) = {} & \mathbf{A}_{\mathrm{r}}(T)\mathbf{e}(t) + \mathbf{A}_{\tau_{\mathrm{r}}}(T)\mathbf{e}(t-\tau) + \big(\mathbf{A}_{\mathrm{d}}(T) - \mathbf{A}_{\mathrm{r}}(T)\big)\mathbf{x}(t) + \big(\mathbf{A}_{\tau_{\mathrm{d}}}(T) - \\ & \mathbf{A}_{\tau_{\mathrm{r}}}(T)\big)\mathbf{x}(t-\tau) - \mathbf{B}(T)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(T)\big(\mathbf{P}(t)\mathbf{e}(t) + \mathbf{g}(t)\big) + \mathbf{z}_{\mathrm{d}}(t) - \mathbf{z}_{\mathrm{r}}(t) \ .\end{aligned} \tag{19}$$

This procedure must be performed in each approximation of the sequence of linear differential equations of (7) until the error signal $\mathbf{e}(t)$, on the ith iteration, tends towards zero.

4 Synchronisation of Chaotic Dynamics in Semiconductor Lasers with Optical Feedback

The dynamics of a semiconductor laser with optical feedback can be represented by using the Lang-Kobayashi equations [3]. These equations describe the effect of weak-to-moderate optical feedback on laser diodes. This set of equations shows the evolution of the complex electric field $E(t)$ and the carrier density $N(t)$. The effect of the optical feedback, which can be generated by an external reflector or mirror, is included in the model by a time-delayed complex electric field term, producing a nonlinear time-delay differential equation.

The optical feedback in lasers diodes can destabilise their dynamics creating chaos. The appearance of chaotic dynamics in semiconductor lasers with optical feedback can be utilised in order to develop private communications systems by exploiting this behaviour. One way of taking advantage of the chaotic behaviour of this laser configuration for communication purposes is to synchronise the signals between two lasers. The synchronisation of semiconductor lasers subject to optical feedback have received a lot of attention due to its potential application for secure communications. This section presents the synchronisation of two unidirectionally coupled external cavity laser diodes by applying the approximating sequence technique and the linear optimal regulator problem.

The dimensionless form of the Lang-Kobayashi rate equations [15] can be written in polar coordinates

$$
\begin{aligned}
\frac{dE(t)}{dt} &= N(t)E(t) + \eta E(t-\tau)\cos(\phi(t-\tau) - \phi(t) - \omega_0\tau_e) \ , \\
\frac{d\phi(t)}{dt} &= \alpha N(t) + \eta \frac{E(t-\tau)}{E(t)} \sin(\phi(t-\tau) - \phi(t) - \omega_0\tau_e) \ , \\
T\frac{dN(t)}{dt} &= P - N(t) - (1 + 2N(t))E(t)^2 \ ,
\end{aligned}
\tag{20}
$$

where $E(t)$ is the amplitude of the electric field, $\phi(t)$ is the corresponding phase and $N(t)$ is the carrier density. The time is normalised to the cavity photon lifetime, τ_p, (so that $t \rightarrow t/\tau_p$). T is the ratio of the carrier and photon lifetimes $T \equiv \tau_s/\tau_p$. The external round trip time is τ_e, which is also normalised to the photon lifetime ($\tau \equiv \tau_e/\tau_p$). $\tau_e = 2L/c$, where L is the distance of the semiconductor laser diode from the mirror and c is the speed of light. P is the dimensionless pumping current above threshold. η is the strength of the feedback. α is the linewidth enhancement factor and ω_0 is the frequency of the solitary laser.

Writing (20) in the form $\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t), \mathbf{x}(t-\tau))\mathbf{x}(t) + \mathbf{A}_\tau(\mathbf{x}(t), \mathbf{x}(t-\tau))\mathbf{x}(t-\tau) + \mathbf{B}(\mathbf{x}(t), \mathbf{x}(t-\tau))\mathbf{u}(t) + \mathbf{z}(t)$, yields

$$
\begin{bmatrix} \dot{E}(t) \\ \dot{\phi}(t) \\ \dot{N}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & E(t) \\ 0 & 0 & \alpha \\ -(1+2N(t))\frac{E(t)}{T} & 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} E(t) \\ \phi(t) \\ N(t) \end{bmatrix} + \tag{21}
$$

$$
\begin{bmatrix} \eta\cos(\phi(t-\tau) - \phi(t) - \omega_0\tau_e) & 0 & 0 \\ \frac{\eta}{E(t)}\sin(\phi(t-\tau) - \phi(t) - \omega_0\tau_e) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E(t-\tau) \\ \phi(t-\tau) \\ N(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{P}{T} \end{bmatrix} ,
$$

these are the equations that are used for the drive and response systems. The controller $\mathbf{u}(t)$ only affects the dimensionless carrier density of the response system in order to minimise the error $\mathbf{e}(t)$. The current P is the only parameter used for achieving synchronisation, which is modulated through the controller $(P/T+\mathbf{u}(t))$.

The parameter values are taken from [16] and then transformed to the dimensionless values using the transformations in [15]. The following values are the same for both systems: $\tau_p=2ps$, $\tau_e=0.1ns$, $\tau\equiv\tau_e/\tau_p\equiv50$ and $T=250$. For the drive system: $P=0.513$ (where $J=1.3J_{th}$), $\alpha=6$, $\eta_d=0.0341$ and $\omega_0=2.2161\cdot10^{15}$rad/s ($\lambda=850$nm). For the response system: $P=0.7695$ (where $J=1.45J_{th}$), $\alpha=3.5$, $\eta_r=0.0417$ and $\omega_0=2.5115\cdot10^{15}$rad/s ($\lambda=750$nm).

In order to accomplish the synchronisation, the chosen weighting matrices are: $\mathbf{Q}=[1,0,0;0,0,0;0,0,0]$, $\mathbf{R}=0.8$, and $\mathbf{F}=diag\{0.1,0.1,0.1\}$. The dimensionless initial conditions for the drive system are: $E_{ic}=\phi_{ic}=N_{ic}=1\cdot10^{-4}$. And for the response system: $E_{ic}=3.529\cdot10^{-4}$, $\phi_{ic}=0.357\cdot10^{-4}$ and $N_{ic}=4.397\cdot10^{-4}$. These initial conditions are used for generating the histories of $E(t)$ and $\phi(t)$, which are obtained by simulating the dynamics of the stand alone laser (without optical feedback). Then, at t_0 the initial values for the drive system are: $E_0=1.1\cdot10^{-3}$, $\phi_0=14.42$ and $N_0=9.31\cdot10^{-2}$. And for the response system: $E_0=1.31\cdot10^{-2}$, $\phi_0=12.67$ and $N_0=0.14$.

The solution of the differential equations is obtained by using the Euler method with a step length of 0.05.

The synchronisation between the drive $E_d(t)$ and response $E_r(t)$ is shown in Fig.1, the error between the drive and response systems is displayed in Fig.2 and the controller used for minimising the error is depicted in Fig.3.

The number of approximations used for synchronising the systems is 90. This number is quite high due to the trade-off between the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ and also because of the accuracy of the synchronisation, the complexity of the system and the time span. Note that the systems are started at different initial conditions and because of the different parameter values, they are not identical.

In the proposed synchronisation method, it is assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback, particularly, in the semiconductor lasers field, in which the only state accessible for measurement is the electric field. Therefore, there is a need of estimating unavailable state variables, which is considered for a future improvement of the method. The estimation of unmeasurable state variables can be achieved by using state observers.

5 Conclusions

This work shows the application of an approximation sequence technique and the linear optimal regulator problem to the synchronisation of two unidirectionally coupled external cavity laser diodes. An enhancement of the method is to make the controller robust to parameter uncertainties and external disturbances. The future employment of the method is considered for encoding messages using the

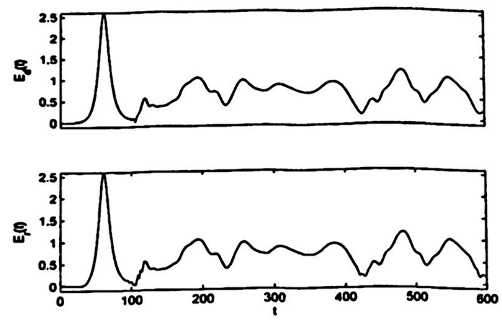

Fig. 1. Synchronisation of the dimensionless electric field amplitude.

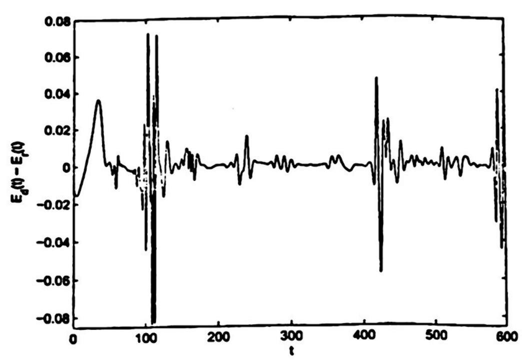

Fig. 2. Error between the drive system and response system.

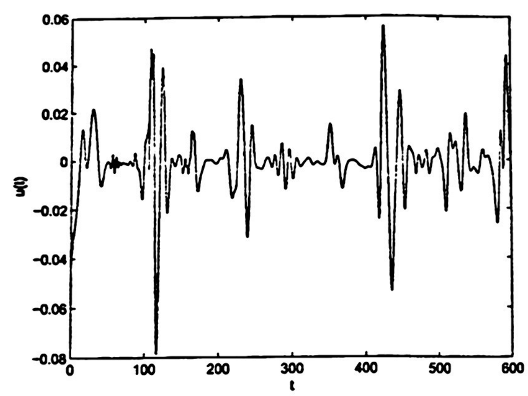

Fig. 3. Controller used for synchronising the drive and response systems.

chaotic output of the master laser, then to transmit the encoded message to the slave laser. The message is masked in the chaotic oscillations and can be decoded after synchronising the master and slave dynamics.

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